ANALYSIS AND SYNTHESIS OF RANDOM MODULATION SCHEMES FOR POWER CONVERTERS

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Abstract

The paper addresses issues in analysis and synthesis of random modulation schemes applied to power converters. After establishing that the proper objects of study are the power spectra of signals, we classify random modulation schemes and present associated spectral formulas, several of which are new. A novel switching scheme based on Markov chains that enables explicit ripple control is also analyzed. Experimental verifications of some of our analytical results are presented. We formulate narrow and wide-band synthesis problems in random modulation, and solve them numerically. Our results suggest that random modulation is very effective in satisfying narrow-band constraints, but has limited effectiveness in meeting broadband power constraints.

1 Introduction

The area of randomized modulation is a topic of current interest in power electronics. While implementation results have been impressive, theoretical analysis has been limited so far. This paper describes the basic theoretical setup needed to address analysis and synthesis problems for a large variety of random switching schemes, and presents representative results.

We tend to agree with [1] that programmed and random switching are complementary techniques, and that by combining them a designer can achieve improved results. The theoretical setup needed to analyze randomized switching schemes is, however, quite different from the deterministic PWM analysis approach. The natural quantity to study in a randomized switching setup is the *power spectrum* (the Fourier transform of the autocorrelation of a signal), and not the harmonic spectrum (i.e. the Fourier transform of the signal). Note that the Fourier transform of a particular realization of a random signal (of arbitrary length) is a random signal itself, i.e. it is a random variable at each frequency. The power spectrum, on the other hand, has much better limiting properties and can be estimated from the available portion of the signal (see for example [2]).

The lack of a proper framework for analyzing random modulation is, in our opinion, the main reason why most references contain only rudimentary analysis, and rely on plausibility arguments. Judging by a sharp increase in the number of papers describing randomized switching implementations (over a dozen in 1992 alone), there exists a definite need for a unifying analysis framework. This will not only make evaluation and verification of different schemes possible, but also point out capabilities and limitations of randomized modulation, which are largely not known at present.

In Section 2 we classify random modulation schemes, and in Section 3 we describe an interpretation of standards for harmonic distortion suitable for random modulation. In Section 4 we present an example of stationary random modulation, and give an analytical result for a random modulation scheme reported earlier in the power electronics literature. In Section 5 we discuss random modulation schemes applicable to inverter operation, and present a generalized formula the for block-version of pulse position modulation. In Section 6 we describe random modulation based on Markov chains, which enables not only power spectral shaping, but also inclusion of time-domain constraints. This type of random modulation could be important in practice, since it offers additional flexibility without being more complicated to implement. In Section 7 we discuss synthesis problems in random modulation, and present numerical results suggesting that random modulation is well suited to meet narrow-band constraints, but much less effective in satisfying wide-band constraints. In this paper we present a sample of our results that are pertinent to random modu-

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lation schemes reported previously in the power electronics literature, and we refer the interested reader to [10] for a detailed discussion.

2 Classification of Random Switching Strategies

To find a common ground for comparisons among different random modulation methods, we concentrate on the switching function, denoted by $q(t)$, which can take only 0–1 values. The power spectrum of variables related to $q(t)$ by linear, time–invariant operations can easily be derived from the power spectrum of $q(t)$. Power spectra for waveforms that are not related to $q(t)$ by such operations can be harder to calculate, but results for certain random modulation schemes are presented in [10].

The main elements characterizing a random modulation scheme are some underlying deterministic nominal switching pattern and the probability laws governing the random dithering of this pattern.

We have to check if the nominal patterns, e.g. duty ratios, vary from one cycle to the next, as they do in inverter operation. The other issue is the time variation of probability densities used to evaluate the dither at each cycle. If the probabilistic structure is constant from cycle to cycle, we call the switching ***stationary***; if it is constant only over a block of cycles (as in inverters) we call it block–stationary. Changes in probabilistic structure that we consider are of Markovian nature, where the probability density used for dither at a cycle depends on a suitably defined state at the beginning of that cycle.

The basic ***analysis*** problem in random modulation is to relate the spectral characteristics of $q(t)$ and other related waveforms in a converter to the probabilistic structure that governs the dithering. The ***synthesis*** problem in random modulation is to design a random switching procedure that minimizes given criteria for power spectra. Practically useful optimization procedures include the minimization of discrete spectral components (denoted as narrow–band optimization in this paper), and the minimization of signal power in a given frequency segment (denoted as wide–band optimization).

3 Performance Specifications

International standards in power electronics are given in terms of Fourier components of a waveform, not in terms of its power spectrum, because they are meant for *periodic* operation. Note however that the power density spectrum of a periodic frequency at each frequency is the square of the magnitude of the corresponding Fourier component. We therefore map the standards to the power spectrum domain by squaring them, and apply the resulting constraints to the non–periodic waveforms obtained by random modulation.

4 A New Result for Asynchronous Modulation

In this section we present an example a of stationary random modulation scheme. This is meant as an illustration, and we refer the interested reader to [10] for an extensive discussion of stationary random modulation. We consider the DC/DC random modulation scheme introduced in [3], without power spectral formulas. In this asynchronous scheme, the lengths of successive cycles T_i are randomized, while the duty ratio is kept fixed. The scheme is different from the simplified version used by the same authors later [4], in which T_i is random, but the duration of the on–state is constant. This latter simplified asynchronous scheme has been analyzed in [5], [6], building on a formula for the power spectrum of a dithered impulse train. The original scheme, when the duty ratio is fixed, is harder to analyze, as two *correlated* dithered impulse trains have to be used. If we denote by $P(f)$ the Fourier transform of the probability density function $p(T)$ used to determine successive cycle lengths T_i, then our derivation shows that the power spectrum of the pulse train is

$$S_q(f) = \frac{2}{(2\pi f)^2}[1 + Re(\frac{P(f)}{1-P(f)}) + +Re(\frac{P^2(\frac{f}{2})}{1-P(f)}) - 2Re(\frac{P(\frac{f}{2})}{1-P(f)})]$$

In Figure 1 we show the calculated spectrum (dotted line), and the estimated spectrum (obtained via Monte–Carlo simulations and Welch's estimation method). Verification issues for power spectral formulas have been addressed in [11].

5 Random Modulation for Inverter Operation

In this section, the nominal on–off pattern is assumed to change from one switching cycle to the next, but repeats periodically over a *block* of cycles. This pattern is then dithered in each cycle using a set of mutually independent trials with statistical structure that remains constant from block to block. Power spectra for line–to–line or line–to–neutral variables in a *three phase* system can be found easily, once the spectrum corresponding to one phase is known.

Consider the case of N basic duty ratios in a block of cycles, with possibly different cycle lengths T_i, and possibly different probability densities (with characteristic functions P_i). Let $U_i(f)$ denote the Fourier transform of the pulse

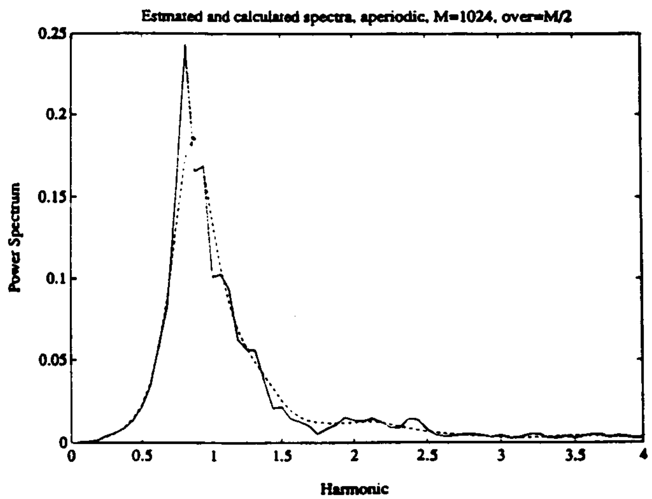

Figure 1: Calculated and Estimated Power Spectra for $q(t)$ in Asynchronous Modulation

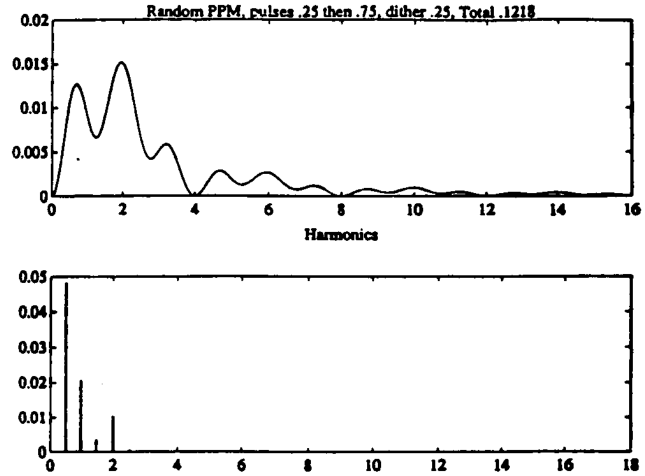

Figure 2: Calculated Spectrum for Block–Stationary Random Modulation Example.

$u_i(t)$ in the i–th cycle of the block (defined with respect to a time origin at the start of the corresponding cycle), and let

$$U(f) = \begin{bmatrix} U_1(f) \\ U_2(f)e^{-i2\pi f T_1} \\ \vdots \\ U_N(f)e^{-i2\pi f \sum_{l=1}^{N-1} T_l} \end{bmatrix}$$

and

$$\hat{U}(f) = \begin{bmatrix} U_1(f)P_1(f) \\ U_2(f)P_2(f)e^{-i2\pi f T_1} \\ \vdots \\ U_N(f)P_N(f)e^{-i2\pi f \sum_{l=1}^{N-1} T_l} \end{bmatrix}$$

Let $\mathbf{1}$ denote an $N \times 1$ vector of ones, and let $\sum_1^N T_i = \bar{T}$. Then the power spectrum of the resulting waveform equals:

$$S_q(f) = \frac{1}{\bar{T}}(\|U\|^2 - \|\hat{U}\|^2) + \frac{1}{\bar{T}^2}\mathbf{1}^T \hat{U}\hat{U}^H \mathbf{1} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{\bar{T}}) \qquad (1)$$

where $\|U\|^2$ is the sum of magnitudes squared of the elements of vector U, and U^H is the complex conjugate transpose (Hermitian) of U. A special case of (1) governs the setup considered in [7].

We present experimental verification for the previous formula next. Experimental circuit comprised of a single phase of a three–phase inverter, and it was driven with a pattern whose randomization was determined by a 68332 microprocessor. The average switching frequency was 250 Hz. All our experiments were meant to verify various theoretical results, and each individual random modulation scheme can be implemented with much simpler hardware. Consider the case $N = 2$, with uniform dither between 0 and $\frac{T_0}{4} = .25$. The basic pulses have $D_1 = .25$ and $D_2 = .75$ and the calculated spectrum is given in Fig. 2. The results shown in Fig. 2 are in close agreement with the experimental results for the same case, which are shown in Fig 3. In this example the discrepancies between theoretical and experimental results for discrete harmonics are under 5%.

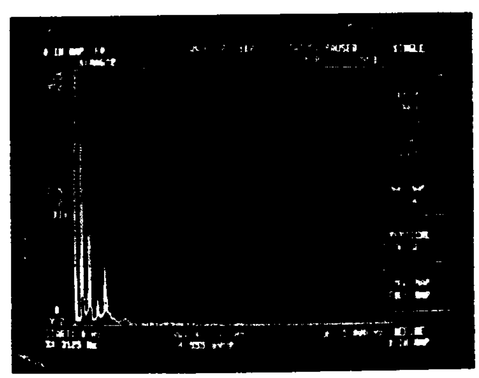

Figure 3: Measured Power Spectrum in Block–Stationary Random Modulation Example.

6 Modulation Based on Markov Chains

In this section we present an example to show the applicability to power electronic systems of a random modulation scheme based on Markov chains. Some analysis results for this class of modulation have been provided by communication theorists [9],[8], but applications in power electronics have not been suggested before. The case of inverter modulation based on Markov chains with different lengths of basic cycles requires results that are not in the communication theory literature either; our results for this scheme are given in [10].

Consider a scheme for DC/DC converters. Suppose we have two kinds of duty ratios available (long L, D=.75, and short S, D=.25). The duty ratios have the desired average of 0.5, but we want to discourage long sequences of pulses of the same kind, thus preventing ripple buildup. We introduce a 4 state Markov chain, corresponding to the following policy. The controller observes the last two pulses and if they are SL or LS, then either of the pulses is fired with probability 0.5. If the pair observed is LL, then an S pulse is applied with probability 0.75 (and an L pulse with probability 0.25). If the pair observed is SS, then an L pulse is applied with probability 0.75 (and an S pulse with probability 0.25).

The theoretical discrete and continuous spectrum corresponding to our example are shown in Fig. 4, where unit frequency corresponds to the switching frequency. The measured power spectrum in the same case is shown in Fig. 5. The circuit used for experimental verification was a down converter, without the output capacitor, and the nominal switching frequency was 10 kHz. Our experimental experience is that random modulation schemes based on Markov chains are not more difficult to implement than the schemes reported earlier in the literature.

The results can be compared with deterministic switching at a constant duty ratio 0.5, in which case only discrete spectrum exists, with a first harmonic of $\frac{1}{\pi^2} = 0.1013$, for example (and subsequent odd harmonics reduced by $\frac{1}{n^2}$). Another meaningful comparison is with a random PWM scheme in which at each trial a random choice is made between duty ratios .25 and .75, independently of previous outcomes. Calculated and measured spectra in this case are shown in Figs.6 and 7. The experimental setup was the same as in the case of random switching governed by a Markov chain.

While the two schemes are quite similar in terms of their power spectra, their time–domain performance is very different. As an example, let us consider the event "five successive long (L) pulses" in both schemes. This event could be of interest, since it is associated with a fairly large net buildup of the local duty–ratio. In the case of independent random PWM, probability of "5 L in a row" is 0.03125.

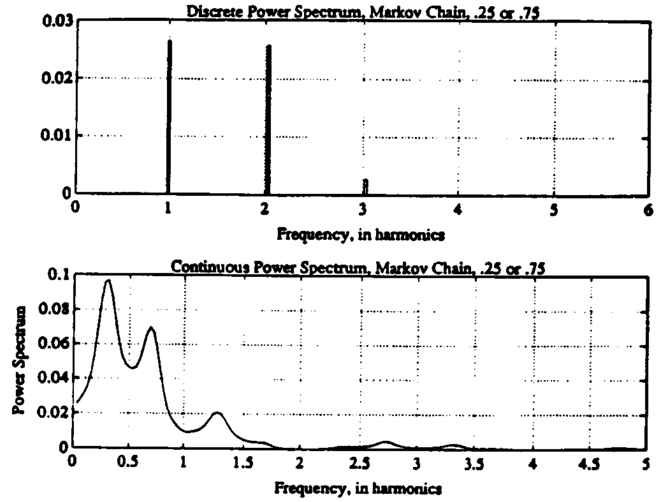

Figure 4: Calculated Power Spectrum of $q(t)$ for the Markov Chain Example

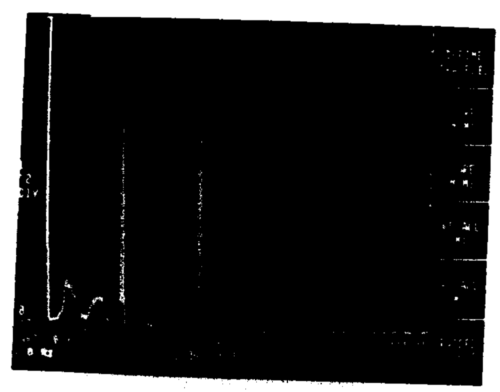

Figure 5: Measured Power Spectrum of $q(t)$ for the Markov Chain Example

In the case of the modulation based on the Markov chain from the example, the same probability equals 0.003125, i.e. it is reduced ten times. These results have been verified both in simulations and in an actual circuit implementation. This example illustrates the power of the Markov chain modulation, which achieves the shaping of the power spectrum, while enabling control of the time–domain waveforms. Other variations are possible. For example, an S pulse could be *required* after an LL pair has been observed in the last two pulses (and symmetrically for an SS pair), thus preventing the occurrence of more than two pulses of the same sort altogether.

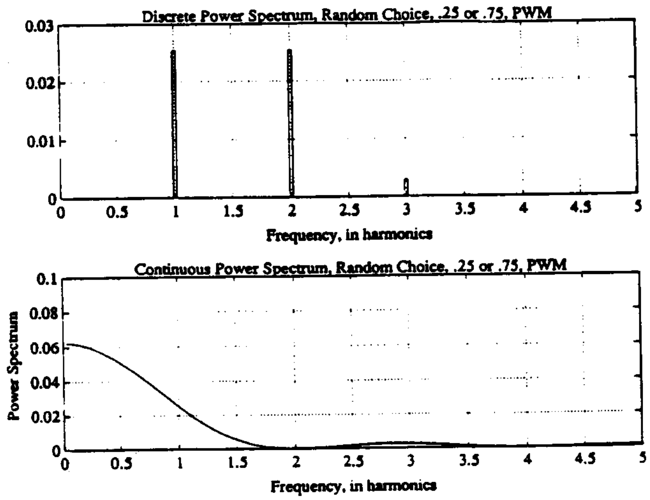

Figure 6: Calculated Power Spectrum of $q(t)$ for the Random PWM Example

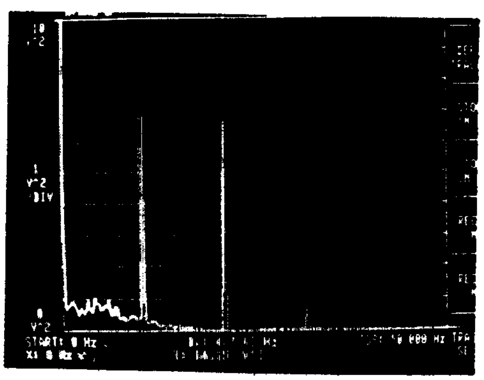

Figure 7: Measured Power Spectrum of $q(t)$ for the Random PWM Example

7 Synthesis Problems

In this section the goal is to explore how effective random modulation is in achieving various performance specifications in the frequency domain. Desirable properties of power spectra are dependent on the particular application. Requirements of particular interest in practice are the following:

- Minimization of one or multiple, possibly weighted, discrete harmonics. This criterion corresponds to cases where narrow–band characteristics of discrete harmonics are particularly harmful, as for example in acoustic noise, or in narrow–band interference.
- Minimization of signal power (integral of the power spectrum) in a frequency segment that is of the order of an integral multiple of the switching frequency. This criterion corresponds to wide–band constraints in military specifications, and it could be of interest for EMI interference problems.

The performance achievable by random modulation depends on the width of the support of the dither probability density, which is in turn dictated by the duty ratio of the nominal switching waveforms. All optimization problems in this section are presented for the case of random PPM. Formulations for other modulation schemes are analogous, and could be specified using the analytic expressions derived in [10]. To streamline the notation, it is assumed that the period of the reference (deterministic) switching waveform is unity, $T_0 = 1$. In that case the power spectrum for random pulse position modulation (PPM) is given by [10]

$$S_q(f) = W(f)[1 - |P(f)|^2 + |P(f)|^2 \sum_{k=-\infty}^{\infty} \delta(f-k)]$$

where the non-negative function $W(f)$ represents the square of the Fourier transform of a rectangular pulse of unit height and width D centered at 0. Note that the k-th discrete harmonic has intensity $|P(k)|^2$. Then a typical narrow–band optimization criterion, which corresponds to the minimization of the sum of discrete harmonics between the l-th and L-th, can be written as

$$J_{l,L}^{NB} = \sum_{k=l}^{L} W(k) \, |P(k)|^2$$

where a weighting function could be absorbed in $W(f)$.

A wide–band optimization criterion used for illustration in this section corresponds to the minimization of the signal power for random PPM in the frequency segment [0 1.5], where the switching frequency is 1. This criterion can be written as

$$J_1^{WB} = \int_0^{\frac{3}{2}} W(f)(1 - |P(f)|^2)df + W(1) \, |P(1)|^2$$

Both the narrow–band criterion J^{NB} and the wide–band criterion J^{WB} are in general nonlinear in $P(f)$; in the PPM case, they are quadratic.

The optimization process, which is performed in the frequency domain, has to generate a ***function*** that satisfies constraints in the time domain. In the case of stationary modulation schemes, the optimization is performed over the space of candidate probability distributions $\mathcal{P}$. In all cases of practical interest these distributions have finite support, which in the example of PPM is constrained to be at most $[0,\ 1-D]$, where D is the nominal (undithered) duty ratio.

If global optimality of solutions to optimization problems in random modulation is needed, then a complete parametrization of the domain $\mathcal{P}$ ***in the frequency domain*** is required. None of the results from Fourier theory known to us establishes a complete parametrization of the set of $P(f)$, even in the absence of constraints in time domain. Thus, for our optimization purposes, it is necessary to construct partial parametrizations of the domain $\mathcal{P}$. Several simple parametrizations of $\mathcal{P}$ were used in our numerical procedures, and together they enable optimization over many probability distributions of interest in implementations. The parametrizations used in the optimization problems here are defined in the time domain as

$$p(t) = \sum_{l=1}^{N} \alpha_l p_l(t)$$

where the p_l are known probability densities ("basis functions") with appropriate finite support, and the α_l are coefficients to be determined in the optimization. The coefficients satisfy $\sum_{l=1}^{N} \alpha_l = 1$, and $\alpha_l \geq 0$. It turns out that optimal probability densities obtained with different basis functions are very similar, and that they yield a very similar performance in terms of the criterion values. This gives some assurance that the choice of basis functions is not critical for the optimization.

The basis functions used to parametrize the domain of candidate probability densities in this paper are:

- Rectangles, dividing the available probability density support ($[0,\ 1-D]$ in the PPM case) into N segments of equal width;
- Hanning windows ("raised cosine" functions), with p_1 and p_N being "half–windows".
- Discrete probability densities of N point masses summing to 1, at fixed or variable locations;
- β–densities, given (on the segment $[0,\ 1-D]$) by the expression

$$p(t) = \frac{1}{B(a,b)} t^{a-1}\ (1-t)^{b-1}$$

where $B(a,b)$ is the normalizing constant

$$B(a,b) = \int_0^{1-D} \sigma^{a-1}(1-\sigma)^{b-1} d\sigma$$

A β–density depends on only two parameters, and it can approximate probability densities having a single maximum or minimum.

Table 1: Narrow-Band Optimization J_{41}^{NB}: Crit. ($\times 10^4$).

Modulation	D=0.1	D=0.5	D=0.9
Undithered	450.0	1250.0	450.0
Point Masses	71.8	0.0	197.0
Hanning	3.3	153.0	232.0
Rectangles	3.5	208.0	242.0
Uniform	3.5	417.0	283.0
Uniform Pt. Masses	255.0	211.0	240.0

Other probability densities used for comparison are the uniform density, and a density comprising N equally spaced probability masses with coefficients $\frac{1}{N}$.

Optimization of the narrow-band criterion $J_{1,41}^{NB} = J_{41}^{NB}$ is considered first, with $N = 4$ basis functions in each parametrization. The criterion values at the numerically computed optimum are given in Table 1. These optimization results are in agreement with the intuition that in cases when there is a large "dithering length" available (i.e., when the nominal duty ratio D is small), then a large reduction of the size of harmonics should be achievable. Though none of our parametrizations for probability densities ends up being superior for all cases, the optimal solutions for different parametrizations are in fact similar. For example, for $D = 0.5$, the optimal rectangle coefficients are [0.5 0 0 0.5], while the optimal peak height ratios for the Hanning basis functions are [0.5 0 0 0.5].

In the case of the duty ratio $D = 0.5$, with equal point masses at 0 and 0.5, the orthogonality of $W(f)$ and $\mid P(f) \mid$ yields 0 as the value of the criterion. This modulation scheme is sometimes referred to as dual modulation. Such extreme effectiveness in the reduction of discrete harmonics is not always a characteristic of random modulation, but it might account for some dramatic improvements reported in implementations.

The criterion J_1^{WB} is considered next, and results obtained for random PPM are given in Table 2. The results suggest a limited effectiveness in reducing the total signal power in a wide frequency band (relative peak heights are given in the third column for the Hanning basis).

These examples suggest that random modulation is in general very effective in reducing the size of discrete components, thus providing a quantitative basis for widespread applications of random modulation to acoustic noise reduction. On the other hand, random modulation is much less

Table 2: Wide-Band Optimization J_1^{WB}, $D = 0.5$: Crit. ($\times 10^4$).

Modulation	Criterion	Coefficients
Undithered	1013	
Point Masses	965	0.5 at 0.16 and 0.34
Hanning	973	[0.33 0.17 0.17 0.33]
Rectangles	973	[0.35 0.15 0.15 0.35]
β–density	975	$a = b = 0.76$

effective in addressing wide–band spectral requirements. Spectral changes introduced by random modulation are mostly localized in frequency. Thus, if the transfer function to the part of circuit under study is known in the frequency band of interest, then an adequate assessment of effects of random modulation is possible.

8 Conclusions

The main motivation for use of random modulation so far has been the possibility of acoustic noise reduction in inverter–based motor drives. It is argued in this paper that random modulation could be beneficial for operation of any power converter. The main benefit from randomized switching strategies in this context is better utilization of the allowable harmonic content of waveforms at the equipment/utility interface. Random modulation is not merely a way to take advantage of present regulations, which have been written for a deterministic switching discipline, but also a flexible approach to solving problems caused by electromagnetic or acoustic noise. To that end, the analytical results reviewed here might serve as an aid to assessment of the potential benefits of random modulation, and as basis for design.

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