Evaluating Piezoelectric Materials and Vibration Modes for Power Conversion

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Abstract—Piezoelectric components offer several potential advantages to power conversion including high power density and efficiency capabilities compared to magnetics at small scales. Converter architectures have been developed for efficient utilization of piezoelectrics, but without fundamental criteria for designing the piezoelectric components themselves. In this paper, we derive figures of merit for the achievable efficiencies and power handling densities of piezoelectric materials and vibration modes based on realistic utilization in a power converter. These figures of merit are likewise accompanied by geometry conditions that serve as guidelines for high-efficiency, high-power-density piezoelectric resonator design. We demonstrate use of these metrics to evaluate commercially-available PZT and lithium niobate materials across seven vibration modes, and we validate the figures of merit and geometry conditions with numerical solutions of converter operation and experimental results. The proposed figures of merit are concluded to be highly representative metrics for the capabilities of piezoelectrics in power conversion, and these capabilities are shown to have favorable scaling properties for converter miniaturization.

Index Terms—piezoelectric resonators, piezoelectric materials, PZT, lithium niobate, dc–dc power conversion

I. INTRODUCTION

Magnetic energy storage elements such as inductors and transformers pose fundamental limits to miniaturization for power electronics; as magnetics scale to smaller sizes, their power density and efficiency capabilities inherently decrease [2], [3]. This motivates exploration of power conversion based on other energy storage technologies that may be more conducive to miniaturization. Piezoelectrics, which store energy in the mechanical compliance and inertia of a piezoelectric material, have very high power density and efficiency capabilities with improved scaling properties to small sizes compared to magnetics [4]. Piezoelectrics also offer the advantages of planar form factors, ease of batch fabrication, and the potential to use the energy storage element itself for electrical isolation.

The promise of power conversion based on only piezoelectric energy storage is evident in magnetic-less converter designs realized in [5]–[10] with single-port piezoelectric resonators (PRs) and in [11]–[18] with multi-port piezoelectric transformers (PTs). In [5], we enumerate practical PR-based converter implementations that achieve high efficiency through strategic utilization of the PR’s resonant cycle, resulting in experimental efficiencies exceeding 99%. Other PR-based converter implementations are demonstrated in [6]-[7], and [8] explores use of high-frequency lithium niobate PRs to achieve high power density. However, criteria for selecting piezoelectric materials and/or designing PRs themselves remain less clear in the context of power conversion. PT structures and design strategies have been investigated in [18]–[27], though these methods are not directly applicable to PRs. Piezoelectric material and loss mechanisms have been reviewed extensively in [28], though primarily in the context of actuation.

In this work, we present figures of merit (FOMs) for piezoelectric materials and vibration modes specifically for use as energy storage in power electronics. Focusing on PRs, we derive FOMs for achievable efficiency and power handling density, which are shown to depend on only material properties assuming a realistic converter control sequence. We demonstrate use of these FOMs to compare commercially-available materials and vibration modes, and we validate the FOMs with both numerical solutions and experimental results. In addition to material and vibration mode selection, these FOM derivations aid PR geometry design and elucidate fundamental power handling scaling properties for PRs in realistic converter implementations.

Since the initial publication of this work in [1], we note that [8] has likewise explored material selection and PR design with an additional focus on fabrication.
Parallel (Perpendicular (+) $\varepsilon_1$) find inductive capacitive

In which bold quantities represent tensors. Parameters for the direct and converse piezoelectric effects [29], [30]:

For thickness mode, $C_p = \varepsilon^2 \frac{d}{l}$. For radial mode parameters, see Appendix A(c).

II. PIEZOELECTRIC RESONATOR MODEL

PRs can be produced in a variety of shapes and electrode patterns, and each configuration has a unique set of compatible vibration modes depending on the PR’s polarization direction, electrode placement, and boundary conditions. To derive power-conversion-based metrics for piezoelectric materials, we focus on the fundamental frequencies of the vibration modes displayed in Fig. 1. These vibration modes can be grouped into two categories that permeate throughout this work: modes for which the applied and induced electric fields are parallel ($||$), and modes for which these fields are perpendicular ($\perp$).

Piezoelectric materials are governed by coupled constitutive relations between mechanical strain ($S$), mechanical stress ($T$), electric field strength ($E$), and electric flux density ($D$) due to the direct and converse piezoelectric effects [29], [30]:

$$ S = s^E T + d^E E $$

$$ D = d^T T + \varepsilon^T E $$

in which bold quantities represent tensors. Parameters for these equations are defined in Tables I and II. Superscripts of $S$, $T$, $E$, or $D$ indicate the respective state held constant during measurement (e.g., $s^E$ indicates compliance at a zero or constant electric field), and $t$ refers to the transpose.

Combined with the Newtonian equation of motion, (1) and (2) reduce to an acoustic wave equation that dictates sinusoidal time- and space-dependent solutions for $u$, $S$, $T$, and $E$: this is described with more detail in Appendix A. With all surfaces in Fig. 1 assumed to be traction-free, the maximum amplitudes of $S$, $T$, and $E$ each occur at the center of the PR.

The frequency $f$ of the acoustic wave propagating through the PR is expressed as

$$ f = \frac{\kappa}{2\pi} v_a = G_f \kappa_o v_a $$

in which $\kappa$ is the wave number (in rad/m) and $v_a$ is the acoustic velocity (in m/s) of the PR material [29]. We also define $\kappa_o$ to be the geometry-normalized wave number (in rad), from which we have extracted the geometry-dependent factor $G_f$. $G_f$ is different for parallel and perpendicular vibration modes and is displayed for each in Table III. For a given material, $G_f$ sets the resonant frequency of a PR design.

The PR’s electrical impedance can be likewise derived from the acoustic wave solution and is shown in Fig. 2 as a function of $f$. The PR exhibits inductive behavior in the region between the resonant ($f_r$) and anti-resonant ($f_{ar}$) frequencies; this region is of most interest to power conversion since inductive loading enables zero voltage switching (ZVS) and
other high-efficiency behaviors [5]. The inductive region spans the following ranges of $\kappa_o$ for parallel ($||$) and perpendicular (+) vibration modes, respectively:

$$\frac{\pi}{2} < \kappa_o < \frac{\gamma_o+\pi}{2} \quad (5)$$

for which factor $\gamma_o$ is displayed in Table III.

The PR impedance characteristic shown in Fig. 2 can be modeled by an equivalent electrical circuit (ie. the Butterworth-Van Dyke model [31]) illustrated in Fig. 3. For excitation of a PR’s fundamental frequency, this model has the parameters shown in Table III, which are generalized to parallel and perpendicular vibration modes [29], [32], [33]. $A$ is the electrode area, $2\ell$ is the distance between the electrodes, and the material parameters of interest are shown for each vibration mode in Appendix A. This electrical model serves as the basis for how we conceptualize the PR’s behavior: $f_r$ corresponds to series resonance between $L$ and $C$, and $f_{ar}$ occurs at parallel resonance between $C_p$ and the series combination of $L$ and $C$. The model’s full derivation for all considered modes can be found in [29], and [1] provides this derivation among others for the length extensional mode.$^1$

III. AMPLITUDE OF RESONANCE MODEL

We employ the circuit model of Section II to analyze the PR’s behavior in a power converter. Because PRs tend to have very high quality factors in the proximity of their resonant frequencies, we assume $i_L$ to be sinusoidal. The amplitude of $i_L$ ($I_L$) then provides insight into the amplitude of the PR’s mechanical resonance, which dictates its mechanical energy storage and loss [5].

A. Assumed Converter Operation

To model PR behavior as utilized in a converter, we assume operation based on either the highest-efficiency step-down switching sequences analyzed in [5]. These sequences are each comprised of six stages: three “connected/zero stages” (ie. stages in which the PR is connected to the source-load system) alternated with three “open stages” (stages in which the PR is open-circuited and $C_p$ charges/discharges through resonance). Further, these switching sequences are constrained for the following high-efficiency behaviors:

1) Resonant “soft” charging/discharging of the PR’s $C_p$ during open stages.

2) Zero-voltage switching (ZVS) of all switches.

3) All-positive instantaneous power transfer during connected stages.

These highest-efficiency switching sequences are capable of constant-efficiency voltage regulation for $\frac{1}{2} \leq \frac{V_{out}}{V_{in}} \leq 1$. One such switching sequence is referred to as $V_{in}$-$V_{out}$: Zero, $V_{out}$ from [5]. Numbers 1-6B designate connected/zero stages (odd) and open stages (even), $V_{in} = 100$ V, $V_{out} = 60$ V, $P_{out} = 4$ W.

B. Model Introduction

The PR’s ideal amplitude of resonance ($I_L$) can be calculated from the total magnitude of charge transferred by $i_L$ during each resonant cycle ($q_{total}$) as illustrated in Fig. 6. This charge transfer calculation depends on the converter’s specific switching sequence and considers both connected/zero stages and open stages for realistic representation [5]. The highest-efficiency sequences discussed in Section III(a) have the following $I_L$, which varies based on operating point:

$$I_L := \frac{\pi}{2} f q_{total} = \pi \left( \frac{P_{out}}{V_{in}} + f C_p V_{in} \right) \quad (6)$$

where $P_{out} = \frac{V_{out}^2}{R_{load}}$ as defined in Fig. 4.
### Table IV

<table>
<thead>
<tr>
<th>Vibration Mode</th>
<th>(I_{L_{\text{max}S0}})</th>
<th>(I_{L_{\text{max}T0}})</th>
<th>(I_{L_{\text{max}E0}})</th>
<th>(I_{L_{\text{max}La}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length Extensional (s)</td>
<td>(\frac{a_{4}}{a_{3}}S_{\text{max}} \sin(\kappa_{o}))</td>
<td>(v_{d}d_{31}T_{\text{max}} \cos\left(\frac{\kappa}{2}\right))</td>
<td>(v_{d}k_{31}^{2}E_{\text{max}} \tan(\kappa_{o}))</td>
<td>(\sqrt{\frac{4Q_{m}k_{33}^{2}}{\pi}})</td>
</tr>
<tr>
<td>Length Extensional (c)</td>
<td>(\frac{a_{4}}{a_{3}}S_{\text{max}} \sin(\kappa_{o}))</td>
<td>(v_{d}d_{33}(1-k_{33}^{2})T_{\text{max}} \cos\left(\frac{\kappa}{2}\right))</td>
<td>(v_{d}k_{33}^{2}E_{\text{max}} \sin(\kappa_{o}))</td>
<td>(\sqrt{\frac{4Q_{m}k_{33}^{2}}{\pi}})</td>
</tr>
<tr>
<td>Thickness Shear (side)</td>
<td>(\frac{a_{4}}{a_{3}}S_{\text{max}} \sin(\kappa_{o}))</td>
<td>(v_{d}d_{15}(1-k_{15}^{2})T_{\text{max}} \cos\left(\frac{\kappa}{2}\right))</td>
<td>(v_{d}k_{15}^{2}E_{\text{max}} \sin(\kappa_{o}))</td>
<td>(\sqrt{\frac{4Q_{m}k_{15}^{2}}{\pi}})</td>
</tr>
<tr>
<td>Thickness Shear (end)</td>
<td>(\frac{a_{4}}{a_{3}}S_{\text{max}} \sin(\kappa_{o}))</td>
<td>(v_{d}d_{15}(1-k_{15}^{2})T_{\text{max}} \cos\left(\frac{\kappa}{2}\right))</td>
<td>(v_{d}k_{15}^{2}E_{\text{max}} \sin(\kappa_{o}))</td>
<td>(\sqrt{\frac{4Q_{m}k_{15}^{2}}{\pi}})</td>
</tr>
<tr>
<td>Thickness Extensional</td>
<td>(\frac{a_{4}}{a_{3}}S_{\text{max}} \sin(\kappa_{o}))</td>
<td>(v_{a}\frac{e_{33}(1-k_{33}^{2})}{\varepsilon_{33}}T_{\text{max}} \cos\left(\frac{\kappa}{2}\right))</td>
<td>(v_{a}k_{33}^{2}E_{\text{max}} \sin(\kappa_{o}))</td>
<td>(\sqrt{\frac{4Q_{m}k_{33}^{2}}{\pi}})</td>
</tr>
<tr>
<td>Contour Extensional</td>
<td>(\frac{a_{4}}{a_{3}}S_{\text{max}} \sin(\kappa_{o}))</td>
<td>(2v_{d}d_{31}T_{\text{max}} \cos\left(\frac{\kappa}{2}\right))</td>
<td>(v_{d}k_{33}^{2}E_{\text{max}} \sin(\kappa_{o}))</td>
<td>(\sqrt{\frac{4Q_{m}k_{33}^{2}}{\pi}})</td>
</tr>
<tr>
<td>Radial</td>
<td>(\frac{a_{4}}{a_{3}}S_{\text{max}} J_{1}(\kappa_{o}))</td>
<td>(2v_{a}d_{31}(1+\theta)\sin(\kappa_{o}))</td>
<td>(v_{d}k_{33}^{2}E_{\text{max}} \sin(\kappa_{o}))</td>
<td>(\sqrt{\frac{4Q_{m}k_{33}^{2}}{\pi}})</td>
</tr>
</tbody>
</table>

Numerical subscripts indicate the tensor components relevant for each mode [29]. For \(I_{L_{\text{max}Lo}}\,\text{we have substituted} H = \left(\frac{P_{\text{loss}}}{A_{f}}\right)_{\text{max}}\). For radial mode, Appendix A(c) contains the definition for \(\Psi\) and series expansions for Bessel functions.

In accordance with the switching sequences discussed in Section III(a), this \(I_{L}\) equation assumes soft charging/discharging of the PR’s \(C_{p}\), ZVS of all switches, and all-positive instantaneous power transfer for high efficiency [5]. Thus, (6) models the PR’s amplitude of resonance for typical converter operation, which we assume throughout this work.

### C. Material Limits

A PR’s maximum amplitude of resonance \((I_{L_{\text{max}}})\) may be determined by its material’s limits for strain \((S_{\text{max}})\), stress \((T_{\text{max}})\), and electric field \((E_{\text{max}})\). Such potential limits include yield stress and strain, as well as coercive field (i.e., the electric field strength at which depolarization occurs). We can derive the relationship between \(I_{L}\) and the PR’s physical states using the constitutive relations (1)-(2) and the equation of motion. This results in the following relationship between \(I_{L}\) and the amplitude of mechanical displacement (\(\Delta\)) as derived in Appendix B:

\[
I_{L} = \kappa AG_{f} \frac{u_{d}d_{d}}{S_{f}} \Delta \sin(\kappa_{o}) \tag{7}
\]

Then, \(S, T\) and \(E\) each can be related to \(I_{L}\) through \(\Delta\). This strategy can be utilized to determine the \(I_{L}\) limits corresponding to maximum strain \((I_{L_{\text{max}S}})\), maximum stress \((I_{L_{\text{max}T}})\), and maximum \(E\) field \((I_{L_{\text{max}E}})\) as demonstrated in Appendix B. It is shown that \(I_{L_{\text{max}S}}, I_{L_{\text{max}T}},\) and \(I_{L_{\text{max}E}}\) have identical geometry terms, so their geometry-normalized values (referred to as \(I_{L_{\text{max}So}}, I_{L_{\text{max}To}}\) and \(I_{L_{\text{max}Eo}}\), respectively) are summarized for each considered vibration mode in Table IV. The lowest-magnitude limit can be considered the geometry-normalized maximum for \(I_{L}\):

\[
I_{L_{\text{max}}} := \frac{I_{L_{\text{max}}}}{AG_{f}} = \min(I_{L_{\text{max}So}}, I_{L_{\text{max}To}}, I_{L_{\text{max}Eo}}) \tag{8}
\]

Thus, the geometry-normalized limit \(I_{L_{\text{max}o}}\) (and which physical limit constrains it) can be determined based on only material parameters and limits. \(I_{L_{\text{max}o}}\) is employed to determine maximum power handling density in Section V.

### D. Areal Loss Density Limit

Thermal management limitations may confine a PR’s amplitude of resonance to lower bounds than its material’s physical limits, so we likewise derive an \(I_{L}\) limit based on areal loss density. Material-dependent losses in the PR include mechanical loss and dielectric loss; since mechanical loss tends to dominate close to resonance [34], we focus on only mechanical loss in the context of this work. We estimate mechanical loss using \(I_{L}\) and \(R\) [5]:

\[
P_{\text{loss}} := \frac{1}{2} I_{L}^{2} R \tag{9}
\]

For an \(I_{L}\) limit based on areal loss density, we assume that most PR heat extraction occurs through a surface with area \(A_{s}\), and that a thermal design can safely accommodate a certain quantity of PR loss per \(A_{s}\). This areal loss density relates to the PR’s operation as follows:

\[
P_{\text{loss}} = \frac{1}{2} I_{L}^{2} R = \frac{1}{2} I_{L}^{2} A_{s} \tag{10}
\]

in which we have extracted geometry parameters from \(I_{L}\) and \(R\), assuming \(A_{s} = A\) for parallel (\(||\)) modes, \(A_{s} = G_{f} A_{l}\) for perpendicular (\(\perp\)) modes as defined in Table VI, and \(R_{o}\) for all modes as defined in Table V (where \(R_{o} = G_{f} R\), in which \(G\) contains all geometry terms). For these assumed surfaces, all geometry terms cancel and (10) shows a direct relationship between areal loss density and \(I_{L_{o}}\). This equation can be rearranged to define a loss-limited maximum for \(I_{L_{o}}\):
## TABLE V

<table>
<thead>
<tr>
<th>Vibration Mode</th>
<th>$G$</th>
<th>$B_o$</th>
<th>$R_o$</th>
<th>FOM$_M$</th>
<th>$\gamma_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length Extensional (s)</td>
<td>$\frac{4b}{3}$</td>
<td>$\varepsilon_T^{s,1}(1 - k^s_3)\frac{m_o}{4\pi}V_{in}^2$</td>
<td>$4Q_m \frac{k^s_1}{1 - k^s_3} \frac{1}{\pi^o_{m_o}} \sqrt{\pi^2 + 8 \frac{k^s_1}{1 - k^s_3}}$</td>
<td>Appendix A(c)</td>
<td></td>
</tr>
<tr>
<td>Length Extensional (e)</td>
<td>$\frac{4}{3}$</td>
<td>$\varepsilon_T^{s,1}(1 - k^e_3)\frac{m_o}{4\pi}V_{in}^2$</td>
<td>$4Q_m \frac{k^e_3}{1 - k^e_3} \frac{1}{\pi^o_{m_o}} \sqrt{\pi^2 + 8 \frac{k^e_3}{1 - k^e_3}}$</td>
<td>Appendix A(c)</td>
<td></td>
</tr>
<tr>
<td>Thickness Shear (side)</td>
<td>$\frac{4}{3}$</td>
<td>$\varepsilon_{\delta}^{s,1}\frac{m_o}{4\pi}V_{in}^2$</td>
<td>$4Q_m k^s_3 \frac{1}{\pi^o_{m_o}} \sqrt{\pi^2 - 8k^2_3}$</td>
<td>Appendix A(c)</td>
<td></td>
</tr>
<tr>
<td>Thickness Shear (end)</td>
<td>$\frac{4b}{3}$</td>
<td>$\varepsilon_T^{s,1}(1 - k^e_3)\frac{m_o}{4\pi}V_{in}^2$</td>
<td>$4Q_m \frac{k^e_3}{1 - k^e_3} \frac{1}{\pi^o_{m_o}} \sqrt{\pi^2 + 8 \frac{k^e_3}{1 - k^e_3}}$</td>
<td>Appendix A(c)</td>
<td></td>
</tr>
<tr>
<td>Thickness Extensional</td>
<td>$\frac{4}{3}$</td>
<td>$\varepsilon_T^{s,1}(1 - k^e_3)\frac{m_o}{4\pi}V_{in}^2$</td>
<td>$4Q_m \frac{k^e_3}{1 - k^e_3} \frac{1}{\pi^o_{m_o}} \sqrt{\pi^2 - 8k^2_3}$</td>
<td>Appendix A(c)</td>
<td></td>
</tr>
<tr>
<td>Contour Extensional</td>
<td>$\frac{4}{3}$</td>
<td>$\varepsilon_T^{s,1}(1 - k^e_3)\frac{m_o}{4\pi}V_{in}^2$</td>
<td>$4Q_m \frac{k^e_3}{1 - k^e_3} \frac{1}{\pi^o_{m_o}} \sqrt{\pi^2 + 8 \frac{k^e_3}{1 - k^e_3}}$</td>
<td>Appendix A(c)</td>
<td></td>
</tr>
<tr>
<td>Radial</td>
<td>$\frac{4}{3}$</td>
<td>$\varepsilon_T^{s,1}(1 - k^e_3)\frac{m_o}{4\pi}V_{in}^2$</td>
<td>$4Q_m \frac{k^e_3}{1 - k^e_3} \frac{1}{\pi^o_{m_o}} \sqrt{\pi^2 + 8 \frac{k^e_3}{1 - k^e_3}}$</td>
<td>Appendix A(c)</td>
<td></td>
</tr>
</tbody>
</table>

Numerical subscripts indicate the tensor components relevant for each mode [29]. From (21), $\kappa_o = \frac{\pi^o_{m_o}}{\pi^o_{m_o}}$ (see Appendix A(c) for radial mode).

$$I_{L_{max,Lo}} = \sqrt{\frac{2}{R_o} \left( \frac{P_{loss}}{A_s} \right)_{max}}$$ (11)

For perpendicular modes in which both non-l dimensions are equal, an areal loss density based on $A_s$ can be scaled to $A_s$ using $\frac{P_{loss}}{A_s} = \frac{G}{2} \frac{P_{loss}}{A}$ (for contour extensional mode) or $\frac{G}{2} \frac{P_{loss}}{A}$ (for radial mode). This assumes a PR design will adhere to $G$, the condition for $G$ corresponding to maximum efficiency as derived in Section IV(a).

$I_{L_{max,Lo}}$ is displayed in Table IV for each mode and can be directly compared to $I_{L_{max,So}}$, $I_{L_{max,To}}$ and $I_{L_{max,Eo}}$ in (8) to determine $I_{L_{max,Lo}}$. It can also be utilized in Section V to calculate loss-limited energy and power handling densities.

### IV. MECHANICAL EFFICIENCY FIGURE OF MERIT

To quantitatively compare piezoelectric materials and vibration modes for power conversion, we first focus on achievable PR efficiency; this has implications for both operating cost and thermal management. Efficiency can be expressed as

$$\eta := \frac{P_{out}}{P_{out} + P_{loss}} = \frac{1 + B_o}{1 + B_o}$$ (12)

Thus, the impact of piezoelectric material properties on efficiency can be examined through loss ratio $P_{loss}/P_{out}$, which is desired to be as low as possible. As discussed in Section III, we focus on only mechanical loss for this FOM.

#### A. Minimum Mechanical Loss Ratio

The mechanical loss ratio of a PR can be expressed as follows, assuming $P_{loss}$ in (9) and $I_L$ in (6):

$$\frac{P_{loss}}{P_{out}} = \frac{1}{2} \frac{fC_p}{V_{in}} R_o = \frac{\pi}{2} \left( \frac{P_{out}}{V_{in}^2} R + 2BR + \frac{V_{in}^2 B_o^2}{P_{out}} R \right)$$ (13)

in which we have substituted

$$B := fC_p$$ (14)

This loss ratio equation has only two operating point parameters ($V_{in}$ and $P_{out}$) and two PR-dependent parameters ($B$ and $R$). We can explicitly separate the PR’s material and geometry properties by extracting all geometry parameters from $B$ and $R$, which have the same lumped geometry term $G$ (as reciprocals), leaving only the material-dependent $B_o$ and $R_o$, respectively. Thus, (13) can be rewritten as

$$\frac{P_{loss}}{P_{out}} = \frac{\pi}{2} \left( \frac{P_{out}}{V_{in}^2} R + 2BR_o + \frac{V_{in}^2 B_o^2}{P_{out}} R \right)$$ (15)

where

$$G := G_f A = B \frac{R_o}{R}$$ (16)

These parameters are displayed for each vibration mode in Table V. It should be noted that we treat $Q_m$ as a material property provided by the manufacturer for the purposes of this study, though $Q_m$ may realistically vary from this value based on PR shape, vibration mode, and mounting structure.

To derive the minimum mechanical loss ratio, we assume that the designer has the flexibility to choose the PR’s geometric dimensions. The loss ratio equation reached in (15) is a second-order equation with respect to $G$ as illustrated in Fig. 7(a). Minimizing (15) with respect to $G$ reveals the following $G$ condition (denoted by $\hat{G}$) and corresponding minimum mechanical loss ratio:

$$\hat{G} = \frac{P_{out}}{V_{in} B_o} = \frac{V_{out}^2}{V_{in} B_o R_{load}}$$ (17)

$$\Rightarrow \left( \frac{P_{loss}}{P_{out}} \right)_{min} = 2\pi^2 B_o R_o$$ (18)

The condition in (17) cancels the operating point, load, and PR geometry parameters in the loss ratio equation (18), so the minimum achievable mechanical loss ratio for a PR depends on only its material properties (i.e., quantities assumed in Table II). Taking the inverse of (18), the following unitless factor can be therefore considered a mechanical efficiency FOM (FOM$_M$) for PR materials and vibration modes:

$$\text{FOM}_M := \frac{1}{2\pi^2 B_o R_o}$$ (19)

FOM$_M$ is desired to be as large as possible, and it is summarized in Table V for the considered vibration modes. For an example of this derivation for a single mode, see [1].
B. Geometry Condition

A PR can be designed to achieve its material’s minimum mechanical loss ratio at a nominal operating point by satisfying \( G = \tilde{G} \) with its geometric dimensions. The order of magnitude of \( B_o \) ranges from \( 10^{-8} \) to \( 10^{-6} \) for most piezoelectric materials, which requires \( \tilde{G} \gg 1 \) in (17) for most power conversion applications. Thus, \( \tilde{G} \) often dictates \( l \) to be the shortest geometric dimension, resulting in primarily planar PR designs. Parallel vibration modes can be particularly advantageous for satisfying large \( \tilde{G} \) values in that both the numerator \( (A) \) and denominator \( (l^2) \) have squared length dimensions. Mode configurations for which \( l \) is not the shortest dimension (length extensional and thickness shear, each with end electrodes) have limited practicality; their relative dimension assumptions can only be maintained for very high \( \frac{f_{\text{in}}}{f_{\text{out}}} \).

C. Operating Frequency

Though \( B_o \) and \( R_o \) have no geometry dependence, \( B_o \) depends on \( \kappa_o \), which spans the ranges of (4) and (5) for the inductive region shown in Fig. 2. Although the switching sequence described in Section III(a) naturally spans this inductive region, frequency is not an independent control variable. To maintain the high-efficiency behaviors described in Section III, the exact operating frequency (and therefore exact \( \kappa_o \)) is dictated by the operating point [5].

Thus, we determine the frequency and \( \kappa_o \) that correspond to the minimum loss ratio operating point in (17)-(18) for use in the FOMs of Sections IV and V (though this is also useful for broader converter design). Inserting \( \tilde{G} \) into (6) reveals both addends of \( I_L \) to be equal for this condition, which implies that the PR’s charge transfer is split evenly between connected/zero stages (resonating at \( f_r \) with period \( \frac{1}{f_r} \)) and open stages (resonating at \( f_{ar} \) with period \( \frac{1}{f_{ar}} \)). Thus, both stage types require half of their respective periods (for each transferring \( \frac{1}{2}Q_{\text{total}} \), so the total period adds to \( \frac{1}{2} = \frac{1}{2f_r} + \frac{1}{2f_{ar}} \). This reveals the highest-efficiency operating frequency to be the harmonic mean of \( f_r \) and \( f_{ar} \):

\[
\bar{f} := \frac{2f_rf_{ar}}{f_r + f_{ar}}
\]

which has the following geometry-normalized wave number (also derived for radial mode in Appendix A(c)):

\[
\tilde{\kappa}_o := \frac{\pi \gamma_o}{\pi + \gamma_o}
\]

Thus, assuming the switching sequence constraints listed in Section III(a) are met, this is the operating frequency for which (17) is satisfied and the minimum loss ratio occurs. This is the operating frequency assumed for the remainder of this work.

V. POWER DENSITY FIGURES OF MERIT

A second point of comparison for piezoelectric materials and vibration modes is achievable power handling density, which poses a boundary for converter miniaturization. Useful power density metrics must consider how the PR is to be utilized in a converter (ie. not just energy storage capability), so we again assume the amplitude of resonance model in Section III. To model the PR’s power handling capability as a function of operating constraints, (6) can be rearranged such that power delivered to the load is the following function of \( I_L \):

\[
P_{\text{out}} = \frac{1}{\pi} V_{in} I_L - C_p f V_{in}^2 = \frac{1}{\pi} V_{in} I_L - B V_{in}^2
\]

With (22), we derive maximum energy and power handling densities considering the physical and loss-density limits for \( I_L \) presented in Sections III(b) and III(c), respectively.

A. Volumetric Energy Handling Density

Volumetric power density is a common metric for converter power handling capability with respect to size. The PR’s volumetric power density can be derived by first dividing (22) by volume, where \( V = 2Al \):

\[
P_{\text{out}} \frac{\text{vol}}{V} = \frac{P_{\text{out}}}{2Al} = G_f \left( \frac{1}{2\pi l} V_{in} I_{L_{\text{maxo}}} - \frac{1}{2\pi l^2} B_o V_{in}^2 \right)
\]

in which \( I_L \) has been set equal to \( I_{L_{\text{max}}} \) and geometry terms have been separated from material properties.

It is evident that a PR’s volumetric power density is directly proportional to \( G_f \), and therefore \( f \). The operating frequency of a converter determines its driving and control requirements, so we elect to normalize (23) to \( f \) for even comparison between PR vibration modes with respect to converter capability. Such normalization results in the following expression for volumetric energy handling density, defined as the quantity of energy delivered to the load in one resonant cycle divided by volume (not to be confused with energy storage density):

\[
E_{\text{out}} \frac{\text{vol}}{V} = \frac{P_{\text{out}}}{V} \frac{\text{vol}}{f} = \frac{1}{\kappa_o \gamma_o} \left( \frac{1}{2} V_{in} I_{L_{\text{maxo}}} - \frac{\pi B_o V_{in}^2}{l^2} \right)
\]

We can then maximize this expression by assuming the designer has the flexibility to choose \( l \). Maximizing (24) with respect to \( l \) as illustrated in Fig. 7(b) results in the following condition (denoted \( \hat{l} \)) and maximum volumetric energy handling density:

\[
\hat{l} = 2\pi B_o V_{in} I_{L_{\text{maxo}}}
\]

\[
\Rightarrow \left( E_{\text{out}} \frac{\text{vol}}{V} \right)_{\text{max}} = \frac{I_{L_{\text{maxo}}}}{4\pi \kappa_o \gamma_o B_o}
\]

Thus, all operating point and geometry terms cancel in (26), and the maximum volumetric energy handling density of a PR depends on only its material properties (defined in Table
II). The following can therefore be considered a volumetric energy handling density FOM for direct comparison between PR materials and vibration modes:

\[ FOM_{VED} := \frac{\bar{I}^2_{L_{\text{max}}}}{4\pi \kappa_\alpha v_a B_o} \tag{27} \]

which has units J/m³. \( FOM_{VED} \) is summarized for various PR vibration modes in Table VI; we suggest use of the minimum-loss wave number \( \bar{\kappa}_\alpha \) as detailed in Section IV(c). The corresponding volumetric power density can be calculated by multiplying \( FOM_{VED} \) by a desired operating frequency, though this frequency depends on \( \bar{l} \) for parallel modes².

B. Areal Power Handling Density

In some applications, power handling capability per footprint area may be more useful to the designer than volumetric energy handling density. Areal power density also becomes more relevant for highly-planar PR designs, as often dictated by \( G \) in (17) for maximum efficiency. Like Section III(c), we assume the area of interest \( A_s = A \) for parallel modes and \( A_s = G_{fA} \bar{l} \) for perpendicular modes as displayed in Table VI. Similar to (23), the areal power density can be written as follows with \( I_L = I_{L_{\text{max}}} \) and geometry terms extracted:

\[ \frac{P_{\text{out}}}{A_s} = \frac{1}{\pi} V_{in} I_{L_{\text{max}}} - \frac{1}{l^2} B_o V_{in}^2 \tag{28} \]

This expression can be likewise maximized with respect to \( l \), resulting in the same \( \bar{l} \) condition and the following maximum areal power density:

\[ \bar{l} = \frac{2\pi B_o V_{in}}{I_{L_{\text{max}}}} \tag{29} \]

\[ \Rightarrow \left( \frac{P_{\text{out}}}{A_s} \right)_{\text{max}} = \frac{I^2_{L_{\text{max}}}}{4\pi^2 B_o} \tag{30} \]

²For parallel modes, maximum volumetric power density occurs for \( l = \frac{1}{4}\bar{l} \), though with diminishing returns with respect to frequency for \( l < \bar{l} \). The volumetric power density at \( l = \frac{1}{4}\bar{l} \) is \( \approx 18.5 \% \) greater than that at \( l = \bar{l} \).

Like the maximum volumetric energy handling density in (26), the maximum areal power density for a PR depends on only its material properties. Thus, (30) serves as an areal power handling density FOM for PR materials and vibration modes:

\[ FOM_{APD||} = \frac{I^2_{L_{\text{max}}}}{4\pi^2 B_o} \tag{31} \]

with units W/m². \( FOM_{APD} \) equals \( FOM_{VED} \) scaled by \( \frac{\kappa_\alpha v_a}{\bar{l}} \) and is likewise summarized in Table VI for each considered vibration mode. It should be noted that \( FOM_{APD} \) assumes the area \( A_s \) corresponds to the PR’s relevant footprint, though the validity of this assumption depends on the PR’s shape, vibration mode, and mounting structure. For contour and radial modes (i.e. modes for which both non-l dimensions are equal), \( FOM_{APD} \) may be respectively scaled by \( \frac{1}{\bar{l}} \) and \( \frac{\pi}{G} \) for representative areal power densities based on \( A_s \), assuming a PR design will adhere to the \( G \) condition for maximum efficiency. Since \( G \) varies by operating point, this scaled \( FOM_{APD} \) for such configurations may introduce dependence on operating point information.

VI. MATERIAL AND VIBRATION MODE COMPARISON

Equipped with FOMs for mechanical efficiency and power density, we now demonstrate use of these FOMs to evaluate piezoelectric materials and vibration modes for power conversion. We first compare relative capabilities of commercially-available variants of hard PZT, the most widely-utilized piezoelectric material for sensing, actuation, transduction, and energy harvesting applications. This is followed by a comparison of the seven vibration modes discussed herein for PZT and lithium niobate (LiNbO₃), a second piezoelectric material of emerging interest for power conversion [4], [8]. It should be noted that the results of these studies are meant to be more demonstrative of the FOMs than prescriptive of the materials; the results are sensitive to variation between manufacturers in terms of measurement procedure and/or reporting of material properties (particularly for \( E_{\text{max}} \) and \( Q_m \)). This topic merits a detailed experimental characterization of such materials and vibration modes, which is beyond the scope of this work.

A. Comparing Materials

First, we compare commercial PZT materials based on efficiency and power density for the thickness extensional and thickness shear vibration modes (with side electrodes). We collect and/or calculate the following properties for 30 hard PZT materials from eight different manufacturers: \( Q_m \), \( \varepsilon_{33} \), \( \varepsilon_{11} \), \( k_1 \), \( k_{15} \), and \( v_a \) for each mode (\( v_a = \frac{2N}{\kappa_\alpha} \)), where \( N \) is the manufacturer-provided frequency constant. Sinusoidal amplitude \( E_{\text{max}} \) is conservatively estimated to be 500 V/mm for all materials, and power density capability is assumed to be limited by this \( E_{\text{max}} \).

We use these properties to calculate \( FOM_M \) and \( FOM_{APD} \) for both modes with each material. Fig. 8 displays these results with one FOM on each axis, and the considered materials exhibit wide variation in terms of capability. The thickness extensional mode (denoted by unfilled markers) generally

<table>
<thead>
<tr>
<th>Table VI: Power Density Figures of Merit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vibration Mode</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>Length</td>
</tr>
<tr>
<td>Thickness Shear (side)</td>
</tr>
<tr>
<td>Contour Extensional</td>
</tr>
<tr>
<td>Radial</td>
</tr>
</tbody>
</table>

Numerical subscripts indicate the tensor components relevant for each mode [29]. From (28), \( \bar{\kappa}_\alpha = \frac{\kappa_\alpha}{\bar{l}} \) (see Appendix A(c) for radial mode).
demonstrates higher power density capability given the same $E_{max}$, and thickness shear mode (denoted by filled markers) shows higher efficiency capability given the same $Q_m$.

Figure 8 is further overlaid with areal-loss-density contour lines to help identify thermal management limits, which vary based on design and application constraints. For a given areal loss density limit, all points above the corresponding contour line can be projected downward (in the -y direction, keeping constant $FOM_M$) onto the allowable heat transfer line itself; this new point conveys their maximum power densities given the assumed limit. It can be inferred that many materials would in fact never reach the assumed $E_{max}$ limit without aggressive thermal management; for a given thermal capability, higher power density is instead enabled by a higher $FOM_M$. In the context of this study, this implies that the thickness shear mode enables higher power densities than the thickness extensional mode for practical areal loss density constraints ($\leq 1 \text{ W/cm}^2$).

**B. Comparing Vibration Modes**

We now compare the seven vibration modes analyzed herein for hard PZT and lithium niobate using material properties from [43] and [44], respectively (we assume these properties as provided, though different crystal cuts can be similarly compared). We calculate $FOM_M$, $FOM_{V_E D}$, and $FOM_{A P D}$ for each vibration mode and display them in Table VII. The same $Q_m$ and $E_{max}$ are assumed for all modes of each material; $Q_m$ is assumed to be 2200 for PIC181 and 10000 for LiNbO$_3$ (as reported in the datasheets), and $E_{max}$ is assumed to be 633 V/mm for PIC181 and 7000 V/mm for LiNbO$_3$ (1/3 the coercive field of each material). An areal loss density limit of 1 W/cm$^2$ is also assumed, so the displayed FOMs are based on the lower of these two limits as described in Section III(c). FOMs that have been scaled by $\frac{G}{\pi}$ or $\frac{P}{\pi}$ are marked with asterisks in Table VII; these FOMs assume $V_{in}=100V$ and $P_{out}=10W$. For this operating point, we also display $\hat{G}$ and a corresponding theoretical design for both maximum efficiency and maximum power density (based on $\hat{G}$ and $l$) for each mode; modes for which this theoretical design violates the relative dimension assumptions in Fig. 1 are marked with “N/A”.

Table VII shows the modes of each material to have significantly varying capabilities with respect to mechanical efficiency, volumetric energy handling density, and areal power density. Higher $FOM_M$ is associated with higher $k^2$ (since the same $Q_m$ is assumed for all modes of the same material), and particularly high $FOM_{M}$'s are shown for LiNbO$_3$ shear modes. Most vibration modes have loss-limited power densities as indicated by $FOM_{A P D}$ tracking $FOM_M$ given the assumed areal loss density limit. Perpendicular vibration modes tend to have higher $FOM_{V_E D}$ compared to parallel modes, which implies that perpendicular modes are capable of higher volumetric power densities for a given frequency. Also worth noting is that some modes would have to operate at more than an order of magnitude higher frequency than others to achieve the same volumetric power density.

Differences in geometry requirements are further highlighted by $\hat{G}$ and the displayed theoretical designs. LiNbO$_3$ generally requires an order of magnitude higher $\hat{G}$ than PZT, resulting in more planar-shaped geometries that often require more footprint area and/or higher operating frequencies for similar density. Perpendicular modes likewise require more extreme planar shapes for the same $\hat{G}$ compared to par-
parallel modes (as described in Section IV(b), parallel modes have the advantage that the length dimensions comprising \( G \) are squared). This is less pronounced at lower \( G \) values but more exaggerated for higher \( G \), making parallel modes more footprint-effective for high power applications. As \( G \) approaches infinity, all modes besides length extensional (end electrodes) are capable of designs that meet the relative dimension assumptions in Fig. 1. As \( G \) approaches zero, the set of compatible modes reduces to just the two length extensional modes and thickness shear (end electrodes) mode.

Thus, the most appropriate piezoelectric materials and vibration modes depend heavily on the target application space. In addition to PZT and LiNbO\(_3\), there are numerous other piezoelectric materials that may be similarly evaluated for power conversion using these FOMs.

 VII. FIGURE OF MERIT VALIDATION

To evaluate their utility, we now validate the FOMs derived herein with numerically-obtained periodic steady state solutions (PSSS) of converter behavior and experimental results.

A. Periodic Steady State Solution

The switching sequence described in Section III(a) is assumed to operate in periodic steady state, which implies that the PR’s states cycle through the same trajectories every switching cycle. As described in [5], we can quantify these trajectories in a PSSS, which varies based on circuit model parameters and operating point \((V_{\text{in}}, V_{\text{out}}, \text{and } P_{\text{out}})\). An “exact” PSSS (i.e., considering \( R \) in Fig. 3) can be obtained by numerically solving the set of differential equations that govern the PR’s states during each stage of a switching sequence. A PSSS has no dependence on the amplitude of resonance mode or other derivations herein.

With an exact PSSS, we can extract useful information from the PR’s state trajectories like loss and output power for validating the derived FOMs. As such, we first validate the minimum loss ratio and maximum power densities with PSSS data based on geometry and material data for 572 APC International discrete PR parts listed on [45]; these parts consist of round and rectangular PRs of varying dimensions, spanning nine total materials. For each part, we:

1. Calculate its circuit model parameters (as shown in Table III) for a given vibration mode.
2. Based on its dimensions, determine \( V_{\text{in}} \) and \( P_{\text{out}} \) corresponding to its minimum loss ratio and maximum energy handling density using (17) and (25), assuming \( I_{L_{\text{max}}} \) to be limited by an areal loss density of 1 W/cm\(^2\).
3. For this operating point, solve for the exact PSSS of the PR’s states as detailed in [5], constraining the converter switching sequence for the high-efficiency behaviors corresponding to (6) and \( V_{\text{out}} = 0.55V_{\text{in}} \) \((I_L \text{ is predicted to be independent of } V_{\text{out}} \text{ for } V_{\text{in}} > V_{\text{out}} > 0.5 V_{\text{in}})\).
4. Extract \( P_{\text{out}} \) from the PSSS by integrating the PR’s \( i_L \) trajectory during load-connected stages and then multiplying by \( V_{\text{out}} \cdot f \). Extract \( P_{\text{loss}} \) from the PSSS by integrating the square of the PR’s \( i_L \) trajectory during all stages and then multiplying by \( R \cdot f \).

Ultimately, all discrete parts of the same material yield the same minimum loss ratio, maximum energy handling density, and maximum areal power density for a given vibration mode; these quantities translate directly to FOM\(_M\), FOM\(_{V,ED}\), and FOM\(_{APD}\), respectively. In Table VIII, PSSS-calculated results for the length extensional mode (end electrodes) are compared with their derived estimates and demonstrate very small error for all low-loss-ratio materials. Accordingly, the PSSS validates the following:

- FOM\(_M\), \((\frac{P_{\text{out}}}{P_{\text{in}}}\text{ max}) \text{ min, } \tilde{G}, \text{ and } \tilde{r}_o\).
- FOM\(_{V,ED}\), FOM\(_{APD}\), and \( \tilde{I} \).
- The independence of \( \tilde{r}_o, (\frac{P_{\text{out}}}{P_{\text{in}}}\text{ max}) \text{ min, } \frac{E_{\text{out vol}}}{A_x}\text{ max, } (\frac{P_{\text{out}}}{A_x})\text{ max, } \text{ and } \frac{P_{\text{out}}}{A_x}\) from PR geometry and operating point information.
- The dependence of \( \frac{E_{\text{out vol}}}{A_x}, \frac{P_{\text{out}}}{A_x}, \text{ and } \frac{P_{\text{out}}}{A_x}\) on normalized amplitude of resonance \( I_{L_o}\).

### Table VII

<table>
<thead>
<tr>
<th>Mode</th>
<th>Length Ext. (s)</th>
<th>Length Ext. (s)</th>
<th>Thickness Shear (side)</th>
<th>Thickness Shear (end)</th>
<th>Thickness Extensional</th>
<th>Contour Extensional</th>
<th>Radial</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOM(_M)</td>
<td>63.4</td>
<td>377</td>
<td>63.4</td>
<td>264</td>
<td>( l=240 \mu m, b=160 \mu m )</td>
<td>1.26</td>
<td>4.14</td>
</tr>
<tr>
<td>FOM(_{V,ED})</td>
<td>4160</td>
<td>12900</td>
<td>N/A</td>
<td>( l=77 \mu m, \sqrt{\frac{4}{3}}=910 \mu m )</td>
<td>1810</td>
<td>7360</td>
<td>4160</td>
</tr>
<tr>
<td>FOM(_{APD})</td>
<td>12900</td>
<td>N/A</td>
<td>( l=71 \mu m, b=10 \mu m )</td>
<td>9120</td>
<td>4440</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical Design*</td>
<td>12900</td>
<td>N/A</td>
<td>( l=150 \mu m, \sqrt{\frac{2}{3}}=1.3 \mu m )</td>
<td>198</td>
<td>72.0</td>
<td>4450</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( l=36 \mu m, a=2.8 \mu m )</td>
<td>9120</td>
<td>3.14</td>
<td>5090</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( l=34 \mu m, a=2.5 \mu m )</td>
<td>10200</td>
<td>3.12</td>
<td>4030</td>
<td></td>
</tr>
<tr>
<td>Lithium Niobate</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FOM(_M)</td>
<td>345</td>
<td>1930</td>
<td>345</td>
<td>395</td>
<td>N/A</td>
<td>74.0</td>
<td>224</td>
</tr>
<tr>
<td>FOM(_{V,ED})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FOM(_{APD})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical Design*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Assuming an example operating point of \( V_{\text{in}}=100 \text{V} \) and \( P_{\text{out}}=10 \text{W} \). At this operating point, power density is \( E\)-field-limited only for PZT contour ext. and radial modes.
We further validate the mechanical efficiency FOM experimentally using six of the PR parts considered in Section VII. These PRs consist of APC International’s highest-FOM* materials (841, 844, 880, and 881) in different shapes and sizes as pictured in Fig. 9. These parts are selected to be low in frequency (< 600 kHz) to minimize frequency-dependent loss and potential damping effects due to mounting during validation. With each PR, we perform the following:

1) Plot the PR’s impedance characteristic (i.e., Fig. 2) for a given vibration mode using an impedance analyzer.
2) Estimate $Q_m$, $k$, and $C_p$ based on the impedance characteristic, as detailed in Appendix D.
3) Calculate the minimum loss ratio for the estimated $Q_m$ and $k$ using (18). Calculate the minimum-loss-ratio $P_{out}^{min}$ using (17), assuming $V_{in} = 100$ V and charge-equivalent switch capacitances of 250 pF (requiring 500 pF total to be added to $C_p$) [46].

4) Run the PR in the prototype converter shown in Fig. 10 with the $V_{in}$-$V_{out}$ Zero, $V_{out}$ switching sequence, constrained for the high-efficiency behaviors assumed in Section III(a). This prototype and switching sequence have the topology and waveforms of Fig. 4 and 5, respectively. $V_{in} = 100$ V and $V_{out} = 60$ V, implemented with a constant-voltage load. All switching times are feed-forward and manually tuned; for a given $V_{in}$, $V_{in}$, and $P_{out}$, there is a unique tuning point that satisfies the assumptions of Section III.

5) Sweep through multiple power levels surrounding the calculated $P_{out}^{min}$ to identify the minimum loss ratio and corresponding $P_{in}$, $P_{out}$, and $f$, maintaining the same $V_{in}$ and $V_{out}$ and re-tuning switching times as needed for the high-efficiency behaviors assumed in Section III.

The results of these experiments, along with material, vibration mode, and frequency information for each PR, are displayed in Table IX. The estimated PR loss ratio (or $1/FOM^2$) tracks the trend of the experimental whole-converter loss ratio as visualized in Fig. 11, albeit a slight underestimation. Sources for the observed discrepancies include other circuit
losses (e.g., switch loss), as well as differences between the PR’s small-signal (as characterized) and large-signal (as tested) characteristics. Thus, (18) provides a close approximation of the loss ratio to be expected by a given material and vibration mode, validating the utility of FOM$_M$.

VIII. FUNDAMENTAL SCALING PROPERTIES

PRs have been previously suggested to have advantageous power density and efficiency scaling properties compared to magnetics [4]. Equipped with our FOM derivations, we explore how PR capabilities scale with size for the realistic converter operation assumed for this work.

If a PR is scaled in all three dimensions by linear scaling factor $\alpha$ as postulated for magnetics in [2], its volume scales by $\alpha^3$. In the case of a PR with fixed $I_{Lo}$, maximum volumetric power density scales inversely with $\alpha$ as does operating frequency, both due to $G_f$. However, the minimum loss ratio, the maximum volumetric energy handling density, the maximum areal power density, and the areal loss density have no geometry dependence and therefore remain fixed regardless of $\alpha$. Table X summarizes these characteristics.

Thus, as volume is scaled downward ($\alpha < 1$), maximum volumetric power density increases while minimum loss ratio (and therefore maximum efficiency) stays constant. These are favorable scaling properties for converter miniaturization.

IX. CONCLUSIONS

To evaluate piezoelectric materials and vibration modes for power conversion, we have established FOMs for achievable efficiency (FOM$_M$), volumetric energy handling density (FOM$_VED$), and areal power density (FOM$_APD$) based on realistic PR utilization in a converter. These FOMs depend on only material properties and areal loss density limits (if considered), and they correspond to PR geometry conditions $\hat{G}$ and $\tilde{l}$ for realizing both maximum efficiency and maximum power density in a PR design.

The derived FOM$_M$ depends on only $k^2$ and $Q_m$, and its corresponding geometry condition $\hat{G}$ dictates the relative PR dimensions corresponding to maximum efficiency for a given operating point. Parallel modes are particularly advantageous for satisfying $\hat{G}$ with less-extreme planar shapes. Which vibration modes are compatible with a given operating space is likewise dictated by $\hat{G}$, favoring vibration modes with $l$ as their smallest dimensions for most realistic converter applications. Further, the operating frequency at which maximum efficiency occurs is found to be the geometric mean of the PR’s resonant frequencies and anti-resonant frequencies for the assumed operation.

Table X and other manufacturer-provided material properties. $f$, $P_{out}$, $P_{out}$ vs. $f$, and efficiency are measured during operation of the converter prototype at its minimum loss ratio.

In this table, $Q_m$, $k$, and $C_p$ are obtained by PR characterization. (20), (17), and $f_{av}$ and other manufacturer-provided material properties. $f$, $P_{out}$, $P_{out}$ vs. $f$, and efficiency are measured during operation of the converter prototype at its minimum loss ratio.

**TABLE IX**

<table>
<thead>
<tr>
<th>Part No.</th>
<th>Material</th>
<th>Vibration Mode</th>
<th>$Q_m$</th>
<th>$k$</th>
<th>$C_p$ (nF)</th>
<th>$f$ vs. (20) (kHz)</th>
<th>$P_{out}$ vs. (17) (W)</th>
<th>$P_{out}$ vs. $P_{out}$ (PR only)</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1817</td>
<td>841</td>
<td>Length Ext. (s)</td>
<td>700</td>
<td>.32</td>
<td>.993</td>
<td>56.8</td>
<td>56.7</td>
<td>.059</td>
<td>.085</td>
</tr>
<tr>
<td>1105</td>
<td>841</td>
<td>Thickness Ext.</td>
<td>2500</td>
<td>.31</td>
<td>.602</td>
<td>605</td>
<td>596</td>
<td>4.0</td>
<td>6.6</td>
</tr>
<tr>
<td>2040</td>
<td>880</td>
<td>Contour Ext.</td>
<td>1600</td>
<td>.52</td>
<td>1.64</td>
<td>81.1</td>
<td>80.9</td>
<td>1.6</td>
<td>1.7</td>
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<tr>
<td>1553</td>
<td>841</td>
<td>Radial</td>
<td>1700</td>
<td>.58</td>
<td>1.50</td>
<td>80.9</td>
<td>80.9</td>
<td>1.6</td>
<td>1.6</td>
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<tr>
<td>790</td>
<td>844</td>
<td>Radial</td>
<td>1400</td>
<td>.55</td>
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<td>124</td>
<td>122</td>
<td>4.1</td>
<td>4.9</td>
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<tr>
<td>1268</td>
<td>881</td>
<td>Radial</td>
<td>1600</td>
<td>.52</td>
<td>.643</td>
<td>129</td>
<td>129</td>
<td>1.3</td>
<td>1.5</td>
</tr>
</tbody>
</table>

In this table, $Q_m$, $k$, and $C_p$ are obtained by PR characterization. (20), (17), and $f_{av}$ and other manufacturer-provided material properties. $f$, $P_{out}$, $P_{out}$ vs. $f$, and efficiency are measured during operation of the converter prototype at its minimum loss ratio.

**TABLE X**

<table>
<thead>
<tr>
<th>Property</th>
<th>Geometry Dependence</th>
<th>Scaling (fixed $I_{Lo}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\frac{P_{out}}{vol})_{max}$ (23)</td>
<td>$G_f$</td>
<td>$\alpha^{-1}$</td>
</tr>
<tr>
<td>$(\frac{P_{out}}{vol})_{max}$ (26)</td>
<td>none</td>
<td>constant</td>
</tr>
<tr>
<td>$(\frac{P_{out}}{vol})_{max}$ (30)</td>
<td>none</td>
<td>constant</td>
</tr>
<tr>
<td>$(\frac{P_{out}}{vol})_{min}$ (18)</td>
<td>none</td>
<td>constant</td>
</tr>
<tr>
<td>$(\frac{P_{out}}{vol})_{max}$ (10)</td>
<td>none</td>
<td>constant</td>
</tr>
<tr>
<td>$f$ (3)</td>
<td>$G_f$</td>
<td>$\alpha^{-1}$</td>
</tr>
</tbody>
</table>

To evaluate piezoelectric materials and vibration modes for power conversion, we have established FOMs for achievable efficiency (FOM$_M$), volumetric energy handling density (FOM$_VED$), and areal power density (FOM$_APD$) based on realistic PR utilization in a converter. These FOMs depend on only material properties and areal loss density limits (if considered), and they correspond to PR geometry conditions $\hat{G}$ and $\tilde{l}$ for realizing both maximum efficiency and maximum power density in a PR design.

**Fig. 11.** Experimental (whole-converter) minimum loss ratio compared to FOM$_M$ for the PRs of Fig. 9, operated in the converter prototype of Fig. 10.
a multiple of $\frac{1}{G}$ for a representative footprint density. Both of these FOMs have the same geometry condition $l$ for maximum power density at a given operating point.

We illustrate the utility of the proposed FOMs to compare the capabilities of 30 PZT-based materials and seven PR vibration modes for PZT and lithium niobate. These materials and vibration modes vary immensely with respect to the FOMs, though higher FOM$_M$ often enables higher power densities for realistic areal loss density limits. With the same $Q_m$ assumed for each mode, the shear modes demonstrate the highest FOM$_M$, particularly for lithium niobate. Perpendicular modes are generally capable of higher FOM$_V$ to meet $G$, Lithium niobate also necessitates more extreme planar dimensions than PZT to meet $G$, requiring more footprint area and/or higher frequency for the same volumetric power density.

We validate these FOMs and their geometry conditions using a periodic steady state numerical solver and experimental results with commercially-available PEs. All PRs of the same material and vibration mode yield the same minimum loss ratio, maximum energy handling density, and maximum areal power density for a given areal loss density, validating the independence of these quantities from PR geometry and operating point information. The proposed FOMs are demonstrated to be highly representative metrics for the achievable efficiencies and power densities of piezoelectric materials and vibration modes, and their corresponding geometry conditions are verified to facilitate both maximum efficiency and maximum power density in a PR design. Further, the displayed FOM values throughout this work attest to the aptitude of piezoelectrics for power conversion in terms of efficiency and power density capabilities, which are shown to scale favorably for converter miniaturization.

ACKNOWLEDGMENT

The authors gratefully acknowledge Yinglai Xia and Jeronimo Segovia-Fernandez with Texas Instruments for insightful discussions pertaining to this work, as well as Prof. Rohan Abeyaratne with MIT for his engaging course on elasticity. The authors likewise acknowledge Joshua J. Piel with MIT for his work on the PSSS solver and experimental prototype board (as part of previous work [5]).

APPENDIX A

PIEZOELECTRIC RESONATOR WAVE SOLUTION

In this Appendix, we derive the PR’s acoustic wave solution for each vibration mode. Piezoelectric materials are governed by the constitutive relations in (1)-(2) and equation of motion

$$\nabla \cdot \mathbf{T} = \rho \ddot{\mathbf{u}},$$

along with strain-displacement definition $\mathbf{S} = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u} \nabla)$ and Gauss’s Law $\nabla \cdot \mathbf{D} = 0$. Together, these equations determine the behavior of the PR states, which varies by vibration mode.

**TABLE XI**

<table>
<thead>
<tr>
<th>Mode</th>
<th>$v_a$</th>
<th>$d$</th>
<th>$a^E$</th>
<th>$a^T$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length Ext.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length Ext.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thickness shear</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(side)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thickness shear</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(end)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thickness</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extensional</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Contour</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extensional</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Measurement condition conversions [29]:

$\varepsilon_{11}^s = (1 - k_2^2)\varepsilon_{11}^E, \quad \varepsilon_{33}^s = (1 - k_2^2)\varepsilon_{33}^E, \quad s^T_{33} = (1 - k_2^2)s^T_{33}, \quad S_{33} = (1 - k_2^2)S_{33}, \quad C_{11} = (1 - k_2^2)C_{11}.$

**Thickness extensional mode is commonly represented by $\varepsilon_{33}, \varepsilon_{33}^T$, and $S_{33}$.**

A. One-Dimensional Stress/Strain Modes

Most vibration modes considered (length extensional, thickness shear, and thickness extensional) can be modeled by one-dimensional stress/strain (i.e., we consider only one tensor component for each PR state in (1)-(2), (32)). For this case, the coupled constitutive relations take the reduced form:

$$S = s^E T + dE$$

(33)

$$D = dT + \varepsilon^T E$$

(34)

To illustrate the PR behaviors for each vibration mode, we adopt a generalized notation for all parameters without indices; the tensor components pertinent to each vibration mode (using Voigt notation) are displayed in Table XI. We likewise refer to a location along the $l$-dimension axis for parallel modes and $a$-dimension axis for perpendicular modes as generally “$P$”.

With the equation of motion (32), (33)-(34) create an acoustic wave equation for mechanical displacement $u$:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = v_a^2 \frac{\partial^2 \mathbf{u}}{\partial x^2}$$

(35)
The PR is assumed to resonate in the proximity of its lowest-frequency vibration mode for traction-free boundaries in Fig. 1. Thus, the acoustic wave solution is sinusoidal in form:

$$u = \Delta \sin(\kappa x) e^{j\omega t} \quad (36)$$

Inserting (36) into the constitutive relations and enforcing traction-free boundaries yields the solutions for $S$, $T$, and $E$; these solutions are displayed for each mode in Table XII. Fig. 12 illustrates the relative amplitudes of $u$, $S$, $T$, and $E$. These states retain similar spatial dependencies across $x$ for each vibration mode, though their specific directions may be different. Moreover, the maximum amplitudes for $S$, $T$, and $E$ each occur at the center of the PR (at $x = 0$).

This wave solution provides means to analyze the PR's mechanical and electrical limits, which we utilize in Section V and Appendix B. Also, the Butterworth-Van Dyke circuit model (Fig. 3) can be derived from $E$ as detailed in [29].

### B. Contour Extensional Mode

For extensional contour mode, we assume the PR to be under plane stress (i.e., we consider only the normal stress components along the two axes perpendicular to the applied $E$ field). We denote these normal stresses with $\sigma_{rr}$ and $\sigma_{yy}$.

$$\sigma_{rr} \approx \frac{1}{2} \Delta (J_0(\kappa r) - J_2(\kappa r)) e^{j\omega t}$$

$$\sigma_{yy} \approx \frac{1}{2} \Delta J_1(\kappa r) e^{j\omega t}$$

To determine the constitutive relations then have the form [29]:

$$T_1 = \frac{1}{s_{11}(1 - \sigma^2)} \left( \frac{\partial u_1}{\partial x_1} + \sigma \frac{\partial u_2}{\partial x_2} \right) - \frac{d_{31}}{s_{11}(1 - \sigma)} E$$

$$T_2 = \frac{1}{s_{11}(1 - \sigma^2)} \left( \sigma \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) - \frac{d_{31}}{s_{11}(1 - \sigma)} E$$

$$D = \frac{d_{31}}{s_{11}(1 - \sigma)} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + \varepsilon_{33}(1 - k_p^2) E$$

With (32), these equations constitute an acoustic wave equation for mechanical displacement $u$. Its solution can be approximated by two separate waves, which we assume to be identical along their respective dimensions [29]:

$$u_1 = \Delta \sin(\kappa x_1) e^{j\omega t} \quad (40)$$

$$u_2 = \Delta \sin(\kappa x_2) e^{j\omega t} \quad (41)$$

#### TABLE XII

<table>
<thead>
<tr>
<th>Length Ext. (s) and Thickness Shear (s)</th>
<th>Length Ext. (e) and Thickness Shear (s)</th>
<th>Thickness Extensional</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$\kappa \Delta \cos(\kappa x) e^{j\omega t}$</td>
<td>$\kappa \Delta \cos(\kappa x) e^{j\omega t}$</td>
</tr>
<tr>
<td>$T$</td>
<td>$\frac{1}{2} \Delta \cos(\kappa x) - \cos(\kappa_o)$</td>
<td>$\frac{1}{2} \Delta \cos(\kappa x) - \cos(\kappa_o)$</td>
</tr>
<tr>
<td>$E$</td>
<td>$\frac{1}{2} \Delta \left( \cos(\kappa x) - \cos(\kappa_o) \right) e^{j\omega t}$</td>
<td>$\frac{1}{2} \Delta \left( \cos(\kappa x) - \cos(\kappa_o) \right) e^{j\omega t}$</td>
</tr>
</tbody>
</table>

#### TABLE XIII

<table>
<thead>
<tr>
<th>Planar Vibration Mode Derivations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contour Extensional</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>$S$</td>
</tr>
<tr>
<td>$T$</td>
</tr>
<tr>
<td>$E$</td>
</tr>
</tbody>
</table>

### APPENDIX B

Also, the Butterworth-Van Dyke circuit model (Fig. 3) can be derived from $E$ as detailed in [29].
Expansion Around Radial Mode Expression

\[ T_1(\kappa_0) = 0.776 - 1.525(\kappa_0 - 2) - 0.789(\kappa_0 - 2)^2 \]

\[ J_0(\kappa_0) = 0.224 - 0.577(\kappa_0 - 2) - 0.0322(\kappa_0 - 2)^2 \]

\[ J_1(\kappa_0) = 0.577 - 0.645(\kappa_0 - 2) - 0.2(\kappa_0 - 2)^2 \]

\[ J_2(\kappa_0) = 0.353 - 0.224(\kappa_0 - 2) - 0.0560(\kappa_0 - 2)^2 \]

Inserting (40) and (41) into (37) and (38) and enforcing the boundary conditions of (39)

\[ \frac{q}{q^2} = \frac{D}{D^2} = 0 \]

at \( x = 0 \) for the PR's and \( \frac{\partial v}{\partial r} = 0 \) at \( x = \pm \alpha \) provides the analytical solution shown in Table XIII for the PR's \( S, T, \) and \( E \) states. This solution follows the relative amplitudes illustrated in Fig. 12(b).

C. Radial Mode

For the radial vibration mode, we assume only planar stress components parallel to the electrodes; \( E \) is again applied only in the polarization direction. This time, we adopt a cylindrical coordinate system in which \( r \) is the radial coordinate and \( \theta \) is the hoop coordinate. Thus, the constitutive relations are [29]:

\[ T_{rr} = \frac{1}{s_{11}(1-\sigma)} \left( \frac{\partial u_r}{\partial r} + \frac{\sigma u_r}{r} \right) = \frac{d_{31}}{s_{11}(1-\sigma)} E \]

\[ T_{\theta \theta} = \frac{1}{s_{11}(1-\sigma)} \left( \frac{\sigma}{r} \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) = \frac{d_{31}}{s_{11}(1-\sigma)} E \]

\[ D = \frac{d_{31}}{s_{11}(1-\sigma)} \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) + \varepsilon_{0}(1 - k^2) E \]

Together with the equation of motion (32), these constitute a wave equation for mechanical displacement \( u_r \) with solution

\[ u_r = \Delta J_1(\kappa r) e^{j\omega t} \]

for which \( J_n \) is the Bessel function of first kind and \( n \)th order.

Inserting (45) into (42) and enforcing the boundary condition \( T_{rr} = 0 \) at \( r = a \) yields the analytical solution in Table XIII for the PR's \( S, T, \) and \( E \) states, which follows the illustration in Fig. 12(b) for \( x = r \). In this solution, we have substituted

\[ \Psi = \frac{1}{2} J_0(\kappa_0) - \frac{1}{2} J_2(\kappa_0) + \frac{\sigma}{\kappa_0} J_1(\kappa_0) \]

The geometry-normalized wave numbers corresponding to \( f_r \) and \( f_{ar} \) in radial mode are solutions to the following transcendental equations, respectively [29]:

\[ \kappa_o, \frac{J_o(\kappa_o)}{J_1(\kappa_o)} = 1 - \sigma \]

If needed, Bessel functions \( J_0(\kappa_0), J_1(\kappa_0), \) and \( J_2(\kappa_0) \) can be approximated by series expansions around \( \kappa_0 \); second-order expansions around \( \kappa_0 = 2 \) are shown in Table XIV.

The geometry-normalized wave number that corresponds to the minimum-loss-ratio operating frequency is then:

\[ \kappa_o = 2 \sigma \kappa_0 + \frac{\kappa_0}{\kappa_0} \]

Finally, \( E \) in Table XIII can be reduced to the circuit model of Fig. 3 with the parameters shown in Table XV.

### APPENDIX B

### MAXIMUM IL DERIVATION

In this Appendix, we determine the maximum permissible \( I_L \) based on limits for the PR's \( S, T, \) and \( E \) states. To begin, we derive the relationship between \( I_L \) and the PR's maximum displacement amplitude (\( \Delta \)) using constitutive relations (1)-(2) and our solutions in Tables XII-XIII. For a one-dimensional vibration mode, inserting (33) into (34) yields:

\[ D = \frac{d}{s} \frac{\partial u}{\partial x} + \varepsilon^T (1 - k^2) \]

Integrating across the volume of the PR (i.e., across electrode area \( A \) and distance \( 2l \)) gives:

\[ Q = AG_1 \frac{d}{s} \frac{u(G_f)^{-1}}{s} \varepsilon^T (1 - k^2) v_{p,1} \]

for which \( v_{p,1} \) is the first harmonic approximation of \( v_p \). Finally, taking the time derivative yields:

\[ i_{in} = j \omega AG_1 \frac{d}{s} \frac{u(G_f)^{-1}}{s} \frac{d}{dt} \frac{u(G_f)^{-1}}{s} + C_p \frac{dv_{p,1}}{dt} \]

in which \( i_{in} \) can be considered the current entering the PR through its top terminal as modeled in Fig. 3. For planar modes, this process results in a similar form as (52) for contour extensional mode and the following form for radial mode:

\[ i_{in} = j \omega a - 2 \pi d_{31} \frac{d}{s} \frac{u(G_f)^{-1}}{s} \Delta J_1(\kappa_o) e^{j\omega t} + C_p \frac{dv_{p,1}}{dt} \]

These expressions for \( i_{in} \) correspond to Fig. 3 such that \( I_L \) equals the magnitude of the first term in (52) or (53). \( I_L \) is shown for each operating mode in Tables XII and XIII.

From here, \( S, T, \) and \( E \) can each be related to \( I_L \) through \( \Delta \). As shown in Fig. 12, the maximum \( S, T, \) and \( E \) each occur at \( x = 0 \), so we rearrange their respective equations in Tables XII-XIII and focus on \( x = 0 \) to reach \( \Delta_S, \Delta_T, \) and \( \Delta_E \) as functions of \( S_{max}, T_{max} \) and \( E_{max}, \) respectively (\( S_{max} \) and \( T_{max} \) refer to the maximum principal strains and stresses for planar modes; this requires taking the limit as \( r \rightarrow 0 \) for radial mode). Each \( \Delta \) can then be directly inserted into \( I_L \), resulting in the strain-limited (\( I_{L,\text{maxS}} \)), stress-limited (\( I_{L,\text{maxT}} \)), and \( E \)-field-limited (\( I_{L,\text{maxE}} \)) maximum amplitudes of resonance displayed in Tables XII-XIII. \( I_{L,\text{maxS}}, I_{L,\text{maxT}}, \) and \( I_{L,\text{maxE}} \) all have the same geometry terms (\( AG_f \)), which allows direct comparison of their geometry-normalized quantities.
<table>
<thead>
<tr>
<th>Vibration Mode</th>
<th>Material</th>
<th>$P_{out}$ vs. FOM$_M$</th>
<th>$\kappa_v$ vs. (21) (radians)</th>
<th>$\Delta_{vol}$ vs. FOM$_{V,ED}$ (J/m$^3$)</th>
<th>$\Delta_{out}$ vs. FOM$_{APD}$ (W/cm$^2$)</th>
<th>$P_{loss}$ vs. 1.00 (W/cm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length Ext. (side)</td>
<td>840</td>
<td>0.0602</td>
<td>0.0570</td>
<td>1.607</td>
<td>1.163</td>
<td>112.3</td>
</tr>
<tr>
<td>Thickness Shear (side)</td>
<td>840</td>
<td>0.0108</td>
<td>0.0107</td>
<td>1.379</td>
<td>1.373</td>
<td>803.2</td>
</tr>
<tr>
<td>Thickness Shear (end)</td>
<td>840</td>
<td>0.0093</td>
<td>0.0092</td>
<td>1.810</td>
<td>1.795</td>
<td>1004</td>
</tr>
<tr>
<td>Thickness Extensional</td>
<td>842</td>
<td>0.0243</td>
<td>0.0240</td>
<td>1.483</td>
<td>1.490</td>
<td>195.1</td>
</tr>
<tr>
<td>Contour Extensional</td>
<td>841</td>
<td>0.0096</td>
<td>0.0096</td>
<td>1.483</td>
<td>1.490</td>
<td>485.5</td>
</tr>
<tr>
<td>Radial</td>
<td>840</td>
<td>0.0163</td>
<td>0.0161</td>
<td>2.221</td>
<td>2.092</td>
<td>269.5</td>
</tr>
</tbody>
</table>

**APPENDIX C**

**EXTENDED PSSS VALIDATION RESULTS**

PSSS results validating the length extensional mode (end electrodes) FOMs are shown in Table VIII. Results for all other considered modes are displayed in Table XVI. The energy and power density quantities are validated for all materials based on areal loss density constraint (10), though the practical relevance of considered area $A_s$ in this calculation depends on the specific vibration mode and mounting structure.

**APPENDIX D**

**PR CHARACTERIZATION FOR EXPERIMENTAL VALIDATION**

For experimental validation, we first characterize each PR in order to calculate its FOMs based on physical properties. $Q_m$ can be calculated based on the PR’s 3dB bandwidth at resonance as measured using an impedance analyzer [29]:

$$Q_m = \frac{f_r}{BW_{3dB}}$$  

Further, the effective $k$ can be calculated based on the observed resonant and anti-resonant frequencies:

$$k_{eff} = \sqrt{\frac{f_{ar} - f_r^2}{f_{ar}}},$$  

This enables the following mode-specific $k$ calculations [29]:

$$k^2 = \frac{\pi}{2} \sqrt{1 - k_{eff}^2 \cot^2 \left( \frac{\pi}{2} \sqrt{1 - k_{eff}^2} \right)},$$  

$$k_p = \frac{1}{\frac{\pi}{2} \sqrt{1 - k_{eff}^2 \tan \left( \frac{\pi}{2} \sqrt{1 - k_{eff}^2} \right)}},$$  

$$k_{p, radial} = \frac{(1 - \sigma) J_1(\kappa_{ar}) - \kappa_{ar} J_0(\kappa_{ar}) - 2J_1(\kappa_{ar}) - \kappa_{ar} J_0(\kappa_{ar})}{2J_1(\kappa_{ar}) - \kappa_{ar} J_0(\kappa_{ar})},$$  

$C_p$ and other circuit parameters can be extracted by matching Fig. 3 to the impedance characteristic.

**REFERENCES**


